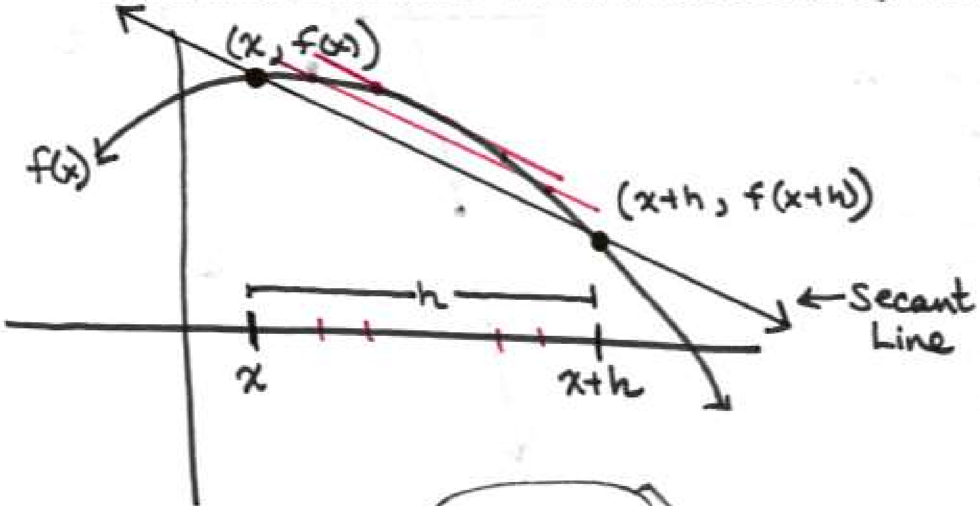
3B 9/25

The Difference Quotient
A First Look at the Derivative

Today we are introduced to the concept with which we will spend our greatest amount of time investigating in Calculus AB—the derivative. Let's draw a picture together.



What does the expression $\frac{f(x+h)-f(x)}{(x+h)-x}$ represent? What does this expression simplify to? $\boxed{\frac{f(x+h)-f(x)}{h}}$

$\frac{f(x+h)-f(x)}{(x+h)-x}$ represents the slope of the secant line pictured above.

As h, the distance between the x – values, x and $(x+h)$, approaches zero, what happens to the secant line?

As $h \to 0$, the secant line becomes a tangent line.

What does the limit $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ represent? the slope of a tangent line that is parallel to the secant line above (slopes are equal).

Suppose $f(x)=-x^2-4x+1$. Find $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

$$\lim_{h\to 0}\frac{-(x+h)^2-4(x+h)+1-(-x^2-4x+1)}{h}$$

$$\lim_{h\to 0}\frac{-\cancel{x^2}-2xh-h^2-\cancel{4x}-4h+\cancel{1}+\cancel{x^2}+\cancel{4x}-\cancel{1}}{h}$$

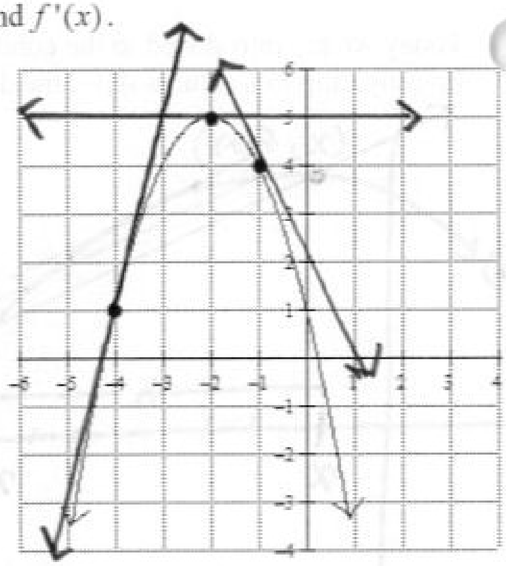
$$\lim_{h\to 0}\frac{\cancel{h}(-2x-h-4)}{\cancel{h}} = -2x-0-4 = \boxed{-2x-4}$$

Your result to the previous limit is defined to be the derivative $\boxed{f'(x)}$ of the function $f(x)$. Now, let's see what this derivative represents in terms of the graph of $f(x)$.

f "prime" of x.

Your result of $\lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$ for $f(x) = -x^2 - 4x + 1$ is a function in terms of x. The graph of $f(x)$ is pictured below. Complete the chart for the indicated x – values and $f'(x)$.

x – value	Value of $f'(x) = -2x - 4$
–4	$f'(-4) = -2(-4) - 4$ $= 8 - 4$ $= 4.$
–2	$f'(-2) = -2(-2) - 4$ $= 4 - 4$ $= 0$
–1	$f'(-1) = -2(-1) - 4$ $= 2 - 4$ $= -2$



Now, use a ruler and draw a tangent line to the graph of $f(x)$ on the grid above at $x = -4$, $x = -2$, and $x = -1$. By investigating the graph, what does it appear that the derivative function $f'(x) = -2x - 4$ represents in terms of the graph at given values of x?

$f'(x)$ is an equation that can be evaluated to determine the slope of any tangent line drawn to the graph of $f(x)$ at any value of x.

Definition of the Derivative and What It Represents Graphically
$f'(x) = \lim\limits_{x \to h} \dfrac{f(x+h) - f(x)}{h}$ $f'(a)$ represents the slope of the tangent line drawn to the graph of $f(x)$ at $x = a$.

Find the equation of the tangent line to $f(x)$ at each of the points below. Then, draw the graphs of the tangent lines on the grid above where $f(x)$ is graphed.

Equation of the tangent line at $x = -4$	Equation of the tangent line at $x = -2$	Equation of the tangent line at $x = -1$
Slope of Tangent = 4 Point of Tangency = $(-4, 1)$ $y - y_1 = m(x - x_1)$ $y - 1 = 4(x + 4)$ $y - 1 = 4x + 16$ $\boxed{y = 4x + 17}$	SOT = 0 POT = $(-2, 5)$ $y - 5 = 0(x + 2)$ $\boxed{y = 5}$	SOT = -2 POT = $(-1, 4)$ $y - 4 = -2(x + 1)$ $y - 4 = -2x - 2$ $\boxed{y = -2x + 2}$

When you hear "**DERIVATIVE,**" you think "**SLOPE OF THE TANGENT LINE.**"

When you hear "**SLOPE OF THE TANGENT LINE,**" you think "**DERIVATIVE.**"

Now that we understand what the derivative of a function represents graphically, let's practice using the limit of the difference quotient $\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)$ to find $f'(x)$ for each of the functions below.

$f(x)=\frac{2}{3}x+3$	$f(x)=\frac{1}{2}x^2-2x+3$
$\lim_{h\to 0}\frac{\frac{2}{3}(x+h)+3-(\frac{2}{3}x+3)}{h}$ $\lim_{h\to 0}\frac{\cancel{\frac{2}{3}x}+\frac{2}{3}h+\cancel{3}-\cancel{\frac{2}{3}x}-\cancel{3}}{h}$ $\lim_{h\to 0}\frac{\frac{2}{3}\cancel{h}}{\cancel{h}}$ $\lim_{h\to 0}\frac{2}{3}=\frac{2}{3}$ $\boxed{f'(x)=\frac{2}{3}}$	$\lim_{h\to 0}\frac{\frac{1}{2}(x+h)^2-2(x+h)+3-(\frac{1}{2}x^2-2x+3)}{h}$ $\lim_{h\to 0}\frac{\cancel{\frac{1}{2}x^2}+xh+\frac{1}{2}h^2-\cancel{2x}-2h+\cancel{3}-\cancel{\frac{1}{2}x^2}+\cancel{2x}-\cancel{3}}{h}$ $\lim_{h\to 0}\frac{\cancel{h}(x+\frac{1}{2}h-2)}{\cancel{h}}$ $x+\frac{1}{2}(0)-2$ $x-2$ $\boxed{f'(x)=x-2}$

Notice that $f'(x)$ for $f(x)=\frac{2}{3}x+3$ was different than $f'(x)$ for $f(x)=\frac{1}{2}x^2-2x+3$. How are they different and why do you suppose this is so?

$f'(x)$ for $f(x)=\frac{2}{3}x+3$ was a constant function but $f'(x)$ for $f(x)=\frac{1}{2}x^2-2x+3$ was a linear function. It appears that the degree of $f'(x)$ is one less than the degree of $f(x)$.

Find $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ for the functions given below to find and use $f'(x)$.

$f(x)=\sqrt{x+2}$ $\lim_{h\to 0} \frac{\sqrt{x+h+2}-\sqrt{x+2}}{h} \cdot \frac{\sqrt{x+h+2}+\sqrt{x+2}}{\sqrt{x+h+2}+\sqrt{x+2}}$ $\lim_{h\to 0} \frac{x+h+2-(x+2)}{h(\sqrt{x+h+2}+\sqrt{x+2})}$ $\lim_{h\to 0} \frac{x+h+2-x-2}{h(\sqrt{x+h+2}+\sqrt{x+2})}$ $\lim_{h\to 0} \frac{1}{\sqrt{x+h+2}+\sqrt{x+2}} = \frac{1}{\sqrt{x+0+2}+\sqrt{x+2}}$ $= \frac{1}{2\sqrt{x+2}}$ $\boxed{f'(x) = \frac{1}{2\sqrt{x+2}}}$	$f(x)=\frac{3}{x+2}$ $\lim_{h\to 0} \frac{\frac{3(x+2)}{x+h+2} - \frac{3(x+h+2)}{x+2}}{h}$ $\lim_{h\to 0} \frac{\frac{3x+6}{(x+2)(x+h+2)} - \frac{3x+3h+6}{(x+2)(x+h+2)}}{h}$ $\lim_{h\to 0} \frac{-3\cancel{h}}{(x+2)(x+h+2)} \cdot \frac{1}{\cancel{h}}$ $\lim_{h\to 0} \frac{-3}{(x+2)(x+h+2)} = \frac{-3}{(x+2)(x+0+2)}$ $= \frac{-3}{(x+2)^2}$ $\boxed{f'(x) = \frac{-3}{(x+2)^2}}$
Find the equation of the line tangent to the graph of $f(x)=\sqrt{x+2}$ at $x=7$. $F(7) = \sqrt{7+2} = \sqrt{9} = 3$ P.O.T $= (7, 3)$ $F'(7) = \frac{1}{2\sqrt{x+2}} = \frac{1}{2\sqrt{7+2}} = \frac{1}{6}$ S.O.T $= \frac{1}{6}$ $y-3 = \frac{1}{6}(x-7)$ $\boxed{y = \frac{1}{6}x + \frac{11}{6}}$	Find the equation of the line tangent to the graph of $f(x)=\frac{3}{x+2}$ at $x=1$. $f(1) = \frac{3}{1+2} = \frac{3}{3} = 1$ P.O.T $= (1, 1)$ $f'(1) = \frac{-3}{(1+2)^2} = \frac{-3}{9} = -\frac{1}{3}$ S.O.T. $= -\frac{1}{3}$ $y-1 = -\frac{1}{3}(x-1)$ $\boxed{y = -\frac{1}{3}x + \frac{4}{3}}$

Using a graphing calculator, graph each of the functions above and the equation of the tangent line that you found to verify your work.

Over the course of this lesson so far, you have found derivatives of several functions and evaluated that derivative at certain x – values. Look back at your work and complete the table below.

Equation of Function, $f(x)$	Equation of Derivative, $f'(x)$	Value of $f'(x)$ at the Indicated value of x	Find the Value of the Limit $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, where a is the value of x.
$f(x)=-x^2-4x+1$	$f'(x)=-2x-4$	$x=-1$ $f'(-1)=-2$ $-2(-1)-4$ $2-4$ $\boxed{-2}$	$\lim_{x\to -1}\frac{-x^2-4x+1-[-(-1)^2-4(-1)+1]}{x-(-1)}$ $\lim_{x\to -1}\frac{-x^2-4x+1-4}{x+1}$ $\lim_{x\to -1}\frac{-x^2-4x-3}{x+1}$ $\lim_{x\to -1}\frac{-1(x^2+4x+3)}{x+1}=\lim_{x\to -1}\frac{-1(x+3)\cancel{(x+1)}}{\cancel{x+1}}=-1(-1+3)$ $=\boxed{-2}$
$f(x)=\sqrt{x+2}$	$f'(x)=\frac{1}{2\sqrt{x+2}}$	$x=7$ $f'(7)=\boxed{\frac{1}{6}}$ $\frac{1}{2\sqrt{7+2}}$ $\frac{1}{6}$	$\lim_{x\to 7}\frac{\sqrt{x+2}-\sqrt{7+2}}{x-7}$ $\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}\cdot\frac{\sqrt{x+2}+3}{\sqrt{x+2}+3}$ $\lim_{x\to 7}\frac{\cancel{x+2-9}}{\cancel{(x-7)}(\sqrt{x+2}+3)}$ $\lim_{x\to 7}\frac{1}{\sqrt{x+2}+3}=\frac{1}{\sqrt{7+2}+3}=\frac{1}{3+3}=\boxed{\frac{1}{6}}$

What inference can you make that explains what the limit $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ represents? The slope of the tangent line drawn to the graph of $f(x)$ at $x=a$.

Complete the table below, stating what each of the indicated limits finds in terms of the derivative of a function, $f(x)$.

Definition of the Derivative	$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$	A function called the derivative that can be used to find the slope of an tangent drawn to $f(x)$.
Alternate Form of the Definition of the Derivative	$\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$	A value that is the slope of the tangent line drawn to the graph of $f(x)$ at $x=a$.